Institute of Media, Information, and Network

Chapter 2 Linear Time-Invariant Systems

Instructor: Hongkai Xiong (熊红凯)

Distinguished Professor (特聘教授)

http://min.sjtu.edu.cn

TAs: Yuhui Xu, Qi Wang

Department of Electronic Engineering

Shanghai Jiao Tong University

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Topic

- 2.0 INTRODUCTION
- 2.1 DISCRETE-TIME LTI SYSTEM:THE CONVOULUTION SUM
- 2.2 CONTINUOUS-TIME LTI SYSTEM:THE CONVOLUTION INTEGRAL
- 2.3 PROPERTIES OF LTI SYSTEM
- 2.4 CASUAL LTI SYSTEMS DESCRIBED BY DIFFERENTAL AND DIFFERENCE EQUATION

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Introduction

- By exploiting the properties of ***superposition*** and ***time invariance***, if we know the response of an LTI system to some inputs, we actually know the response to many inputs

If $$x_k[n] \rightarrow y_k[n]$$

then $$\sum_k a_k x_k[n] \rightarrow \sum_k a_k\, y_k[n]$$

Introduction

- If we can find sets of "**basic**" signals so that
  - We can represent rich classes of signals as linear combinations of these building block signals.
  - The response of LTI Systems to these basic signals are both simple and insightful.
- If we represent input signal as a linear combination of these **basic** signals, then the output is the combination of the responses of such basic signals.
- Candidate sets of basic signal
  - Unit impulse function $\delta(t)/\delta[n]$
  - Complex exponential/sinusoid signals. $e^{j\omega t}/z^n$

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2.1.1 The Representation of Discrete-Time Signals in terms of Impulses

- For example:

  $x[n]=\ldots x[-3]\,\delta[n+3]+x[-2]\,\delta[n+2]+\ldots+x[0]\,\delta[n]+x[1]\,\delta[n-1]+\ldots$

- i.e.: x[n] can be represented as the weighted sum

$$x[n]=\sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

Weight (x[k]) Basic signal (δ[n−k])

2.1.2 Convolution-Sum Representation of LTI Systems

- 1.Assume $\delta[n] \to h[n]$ —— Unit impulse response

$\delta[n-k] \to h[n-k]$ —— Time invariant

and

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]$$

so

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k] \cdot h[n-k] = x[n] * h[n]$$

Convolution-Sum Representation of LTI Systems

- LTI system can be represented by using unit impulse response.
- The output of LTI system is the convolution sum of input and unit impulse response.

Convolution-Sum Representation of LTI Systems

- 2. Convolution sum

$$y[n] = x_1[n] * x_2[n]$$

$$= \sum_{k} x_1[k] \cdot x_2[n-k]$$

$$= \sum_{k} x_2[k] \cdot x_1[n-k]$$

Convolution-Sum Representation of LTI Systems

- 2. Convolution sum
  - ➢Computing method 1-- graphic method
    - Step 1: change variable $n \longrightarrow k$

      $$x_1[n] \longrightarrow x_1[k],\ x_2[n] \longrightarrow x_2[k]$$

    - Step 2: reflect: $x_2[k] \longrightarrow x_2[-k]$
    - Step 3: shift:

      $$x_2[-k] \longrightarrow x_2[n-k]$$

    - Step 4: multiply and sum:

      $$\sum_k x_1[k]x_2[n-k]$$

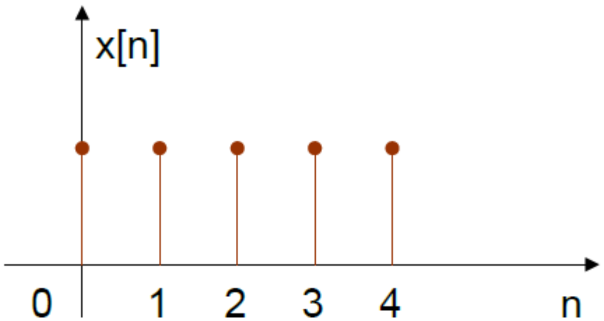
Convolution-Sum Representation of LTI Systems

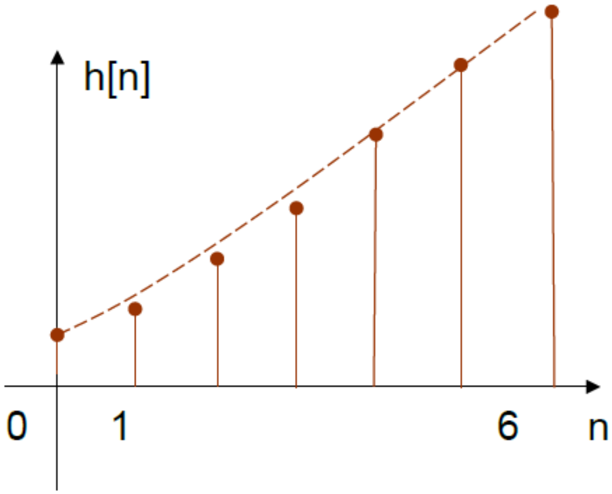
Example:

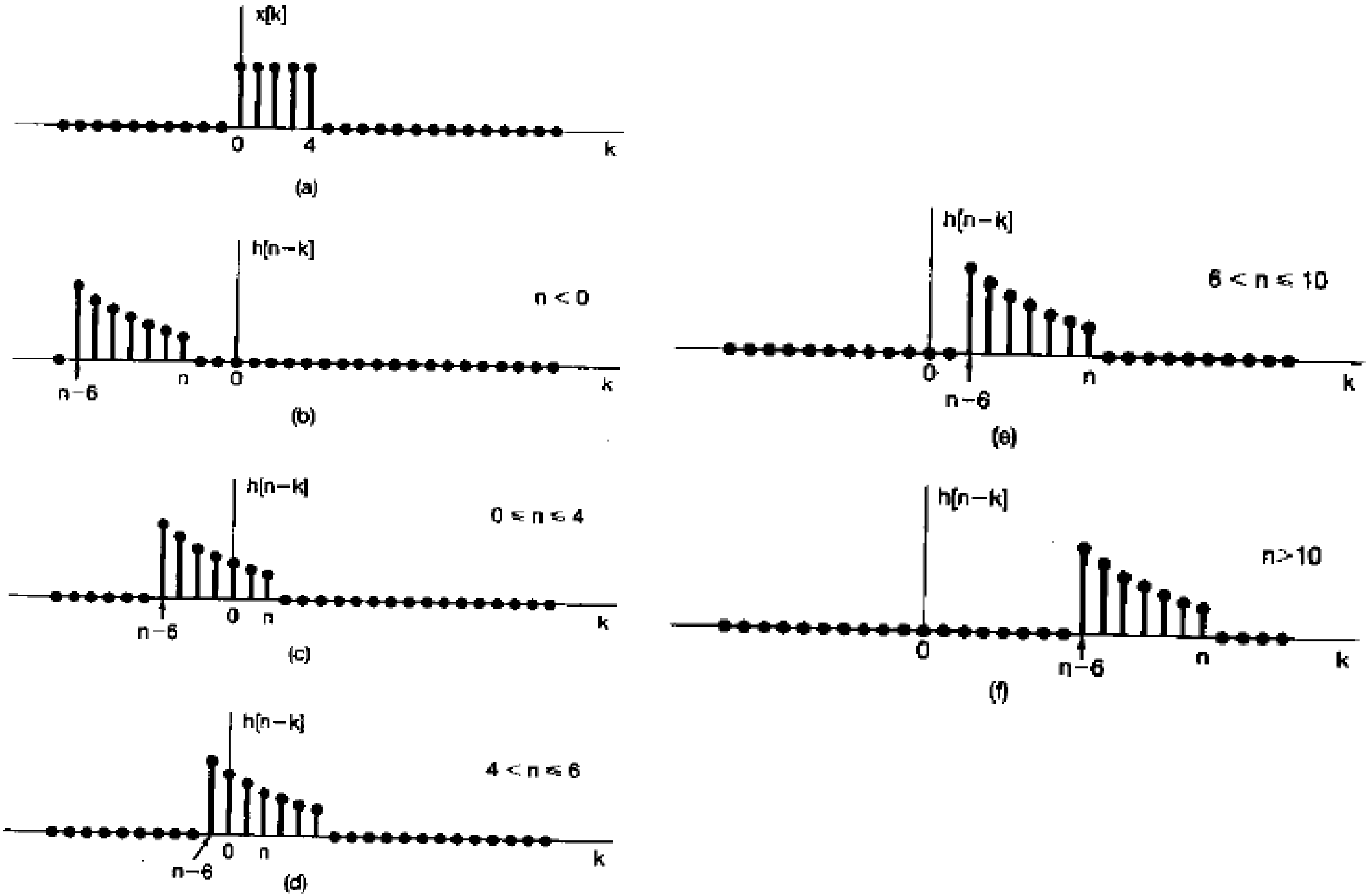
Let $$x[n]=\begin{cases}1 & 0\le n\le 4\\ 0 & others\end{cases}$$

$$h[n]=\begin{cases}a^n & 0\le n\le 6\\ 0 & others\end{cases}$$

Determine $y[n]=x[n]*h[n]$






x[k]
0
4
k
(a)
h[n−k]
n < 0
n−6
n
0
k
(b)
h[n−k]
0 ≤ n ≤ 4
n−6
0
n
k
(c)
h[n−k]
4 < n ≤ 6
n−6
0
n
k
(d)
h[n−k]
6 < n ≤ 10
0
n−6
n
k
(e)
h[n−k]
n>10
0
n−6
n
k
(f)

Convolution-Sum Representation of LTI Systems

- 2. Convolution sum
  - Computing method 2-- the property of $\delta[n]$

$$x[n]*\delta[n]=x[n]$$
$$x[n]*\delta[n-n_0]=x[n-n_0]$$

Example: Let $x_1[n]=u[n+2]-u[n-1]$

$$x_2[n]=2^n(\delta[n]+\delta[n-1]+\delta[n-2])$$

Determine $y[n]=x_1[n]*x_2[n]$

Solution: $x_1[n]=\delta[n+2]+\delta[n+1]+\delta[n]$

$$x_2[n]=\delta[n]+2\delta[n-1]+4\delta[n-2]$$

$$\delta[n-n_1]*\delta[n-n_2]=\delta[n-n_1-n_2]$$

$x_2[n]$ \ $x_1[n]$	$\delta[n+2]$	$\delta[n+1]$	$\delta[n]$
$\delta[n]$	$\delta[n+2]$	$\delta[n+1]$	$\delta[n]$
$2\delta[n-1]$	$2\delta[n+1]$	$2\delta[n]$	$2\delta[n-1]$
$4\delta[n-2]$	$4\delta[n]$	$4\delta[n-1]$	$4\delta[n-2]$

$$y[n]=\delta[n+2]+3\delta[n+1]+7\delta[n]+6\delta[n-1]+4\delta[n-2]$$

Note: only suitable for ***limited length*** sequence.

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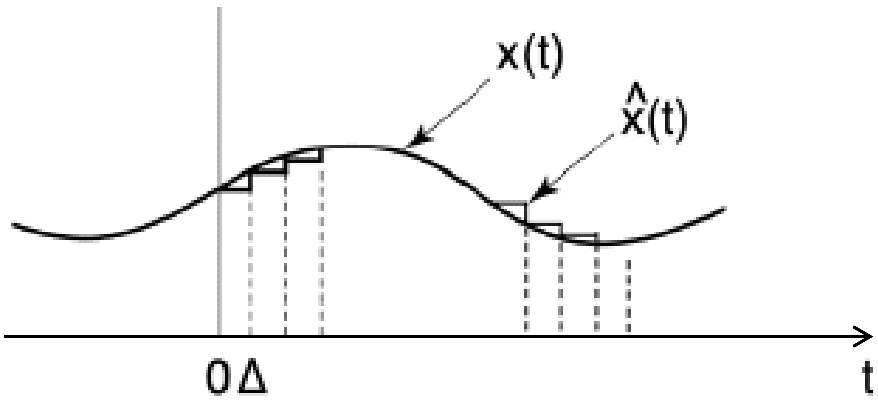
2.2.1 The representation of Continuous-Time Signals

- Approximate a CT signal x(t) as a sum of shifted, scaled pulses
- If

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta} & 0 \le t \le \Delta \\ 0 & \text{others} \end{cases}$$

- then

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$



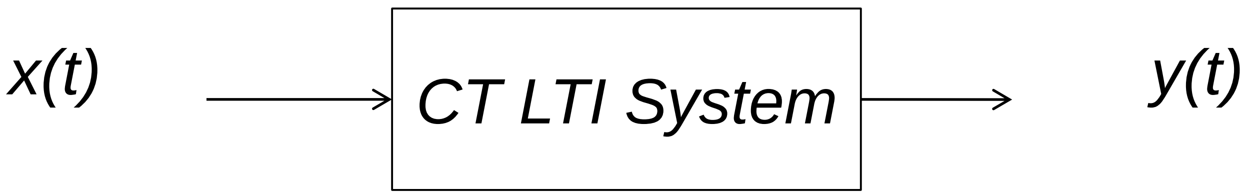
- so

$$x(t) = \lim_{\Delta\to 0}\hat{x}(t) \; = \lim_{\Delta\to 0} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$x(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Weights **Basic Signals**

2.2.2 The Convolution Integral Representation of LTI System



- For a LTI system with the response of h(t) to the unit impulse δ(t)

$$\delta(t) \rightarrow h(t)$$ —— **Unit Impulse Response**

Time-invariance allows

$$\delta(t-\tau) \rightarrow h(t-\tau)$$

The Convolution Integral Representation of LTI System

- Considering the weighted integral of delayed impulse representation of x(t)

$$x(t)=\int_{-\infty}^{\infty}x(\tau)\delta(t-\tau)d\tau$$

- So

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

$$y(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=x(t)*h(t)$$

The Convolution Integral Representation of LTI System

1. A LTI system is completely characterized by its response to the unit impulse ---- ***h(t)***
2. The response *y(t)* to an input CT signal *x(t)* of a LTI system is the convolution of h(t) and x(t)

The Convolution Integral Representation of LTI System

- Convolution Integral

$$y(t) = x_1(t) * x_2(t)$$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \cdot x_2(t-\tau) d\tau$$

$$= \int_{-\infty}^{+\infty} x_2(\tau) \cdot x_1(t-\tau) d\tau$$

The Convolution Integral Representation of LTI System

- Convolution Integral

$$y(t) = x_1(t) * x_2(t)$$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \cdot x_2(t-\tau) d\tau$$

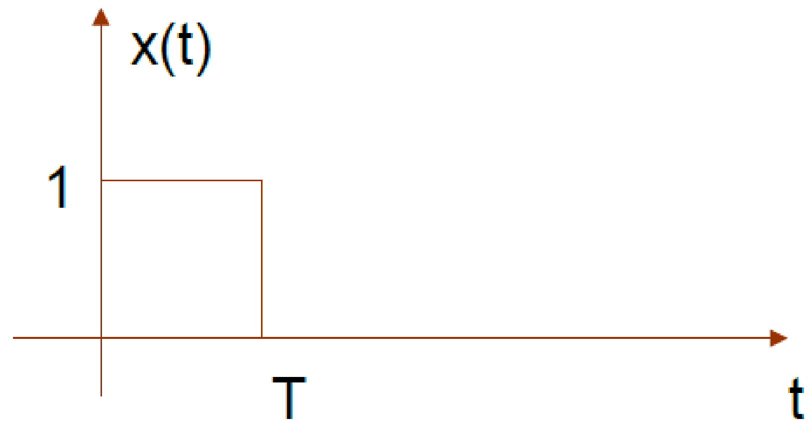
$$= \int_{-\infty}^{+\infty} x_2(\tau) \cdot x_1(t-\tau) d\tau$$

The Convolution Integral Representation of LTI System

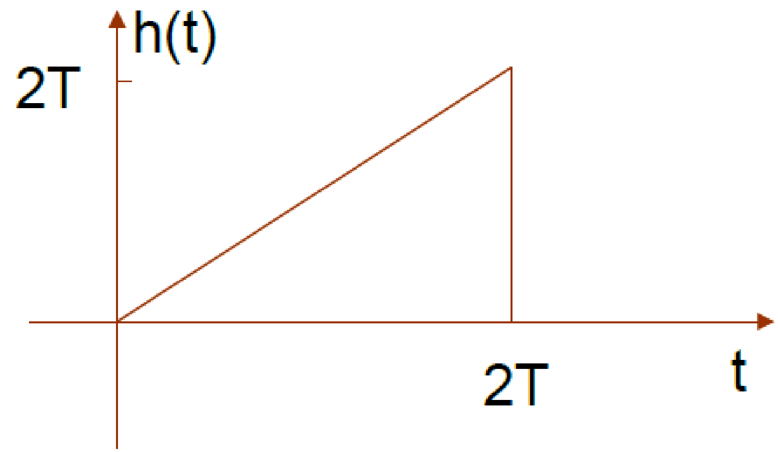
- Method 1 – graphic method
  - Step 1. Replace t with τ for signals $x_1(t)$ and $x_2(t)$, *i.e.* τ *is the independent variable*
  - Step 2. Obtain the time reversal of $x_2(\tau)$
  - Step 3. For the output value at any specific time t, shift $x_2(-\tau)$ with offset t to obtain $x_2(t-\tau)$
  - Step 4. Multiply the two sequences $x_1(\tau)$ *and* x_2t-τ) obtained in Step 1 and Step 3, respectively, and integrate the resulting product from $\tau = -\infty$ to $\tau = \infty$

Example:

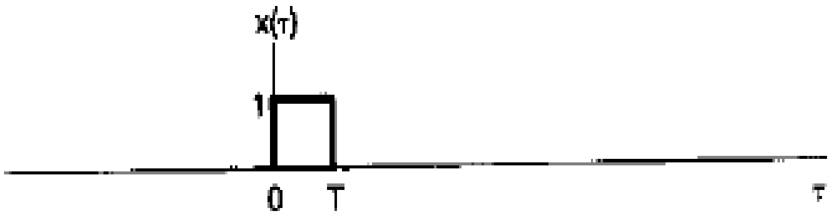
Let $x(t)=\begin{cases}1 & 0<t<T \\ 0 & others\end{cases}$

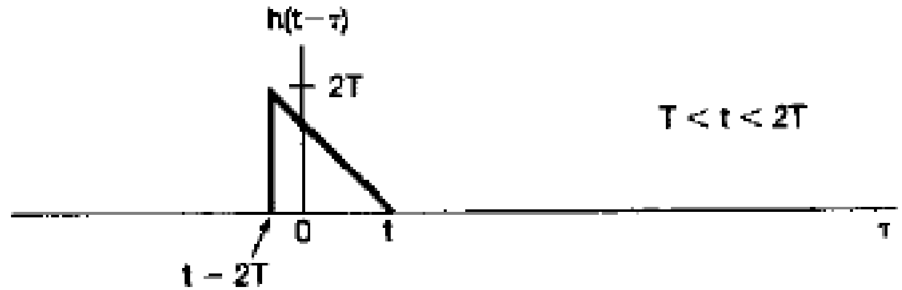


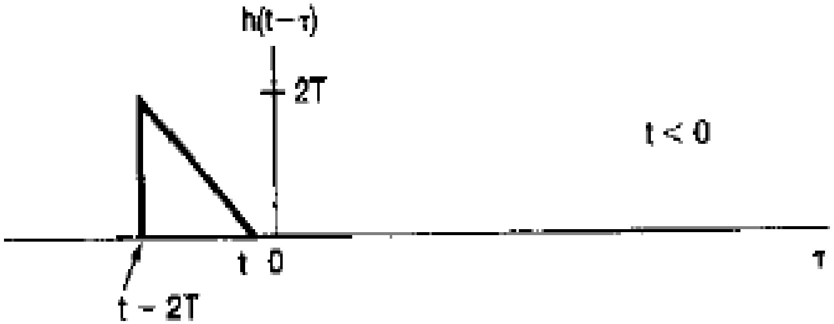
$h(t)=\begin{cases}t & 0<t<2T \\ 0 & others\end{cases}$

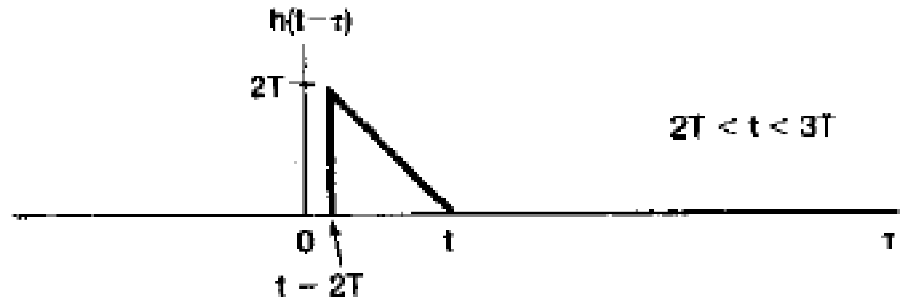


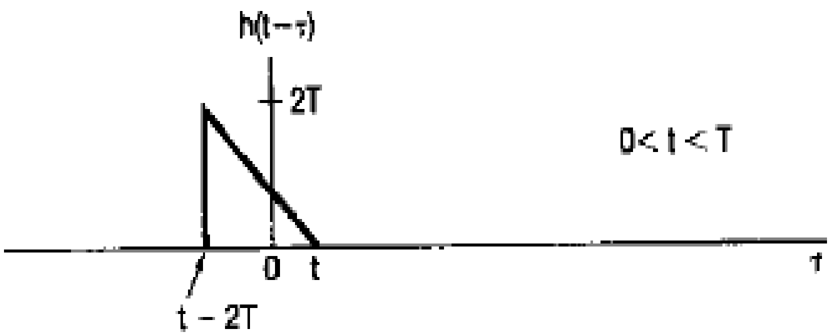
Determine $y(t)=x(t)*h(t)$

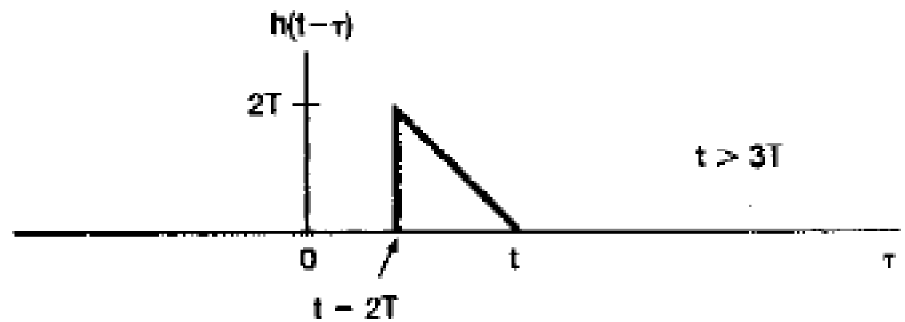

x(τ)
1
0
T
τ


h(t − τ)
2T
T < t < 2T
0
t
τ
t − 2T


h(t − τ)
2T
t < 0
t
0
τ
t − 2T


h(t − τ)
2T
2T < t < 3T
0
t
τ
t − 2T


h(t − τ)
2T
0< t < T
0
t
τ
t − 2T


h(t − τ)
2T
t > 3T
0
t
τ
t − 2T

The Convolution Integral Representation of LTI System

- Method 2 -- exploit the property of δ(t)

$$x(t) * \delta(t) = x(t)$$

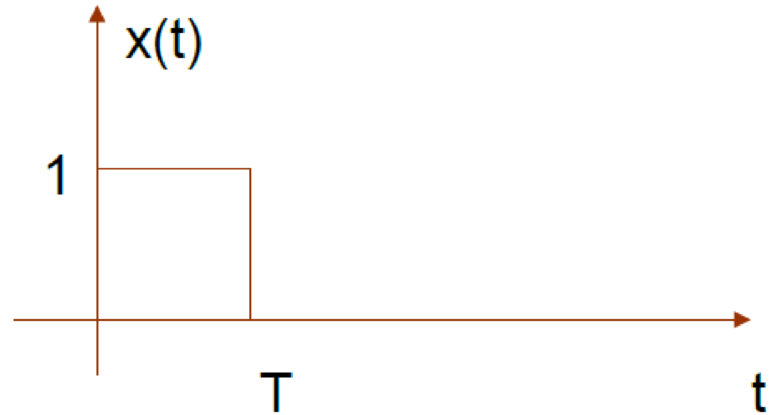
$$x(t) * \delta(t - t_0) = x(t - t_0)$$

If $$y(t) = x_1(t) * x_2(t)$$

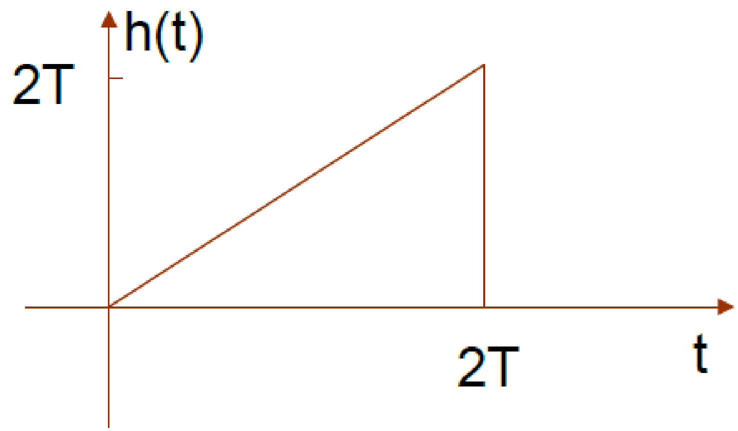
Then $$y^{(i)}(t) = x_1^{(j)}(t) * x_2^{(i-j)}(t)$$

Example:

Let $x(t)=\begin{cases}1 & 0<t<T\\ 0 & others\end{cases}$



$h(t)=\begin{cases}t & 0<t<2T\\ 0 & others\end{cases}$



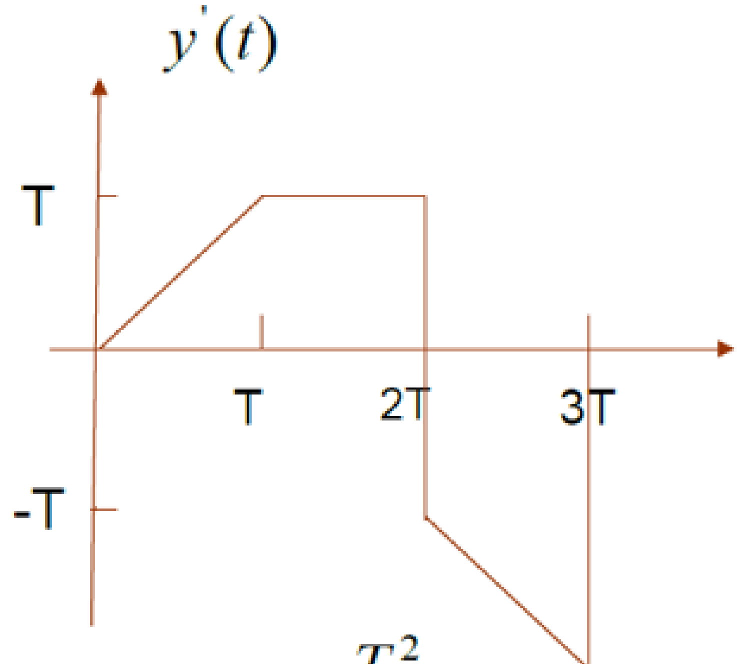
Determine $y(t)=x(t)*h(t)$

$$y'(t) = x'(t) * h(t) \qquad y(t) = \int_{-\infty}^{t} y'(\tau) d\tau$$

$$\because x'(t) = \delta(t) - \delta(t - T)$$

$$\therefore y'(t) = x'(t) * h(t) = h(t) - h(t - T)$$

$$= \begin{cases} t & 0 \le t < T \\ T & T \le t < 2T \\ T - t & 2T \le t < 3T \\ 0 & others \end{cases}$$



$$y(t)=\int_{-\infty}^{t} y^{'}(\tau)d\tau$$

$$while \quad 0 \le t < T$$

$$y(t)=\int_{0}^{t} \tau d\tau = \frac{t^2}{2}$$

$$while \quad T \le t < 2T$$

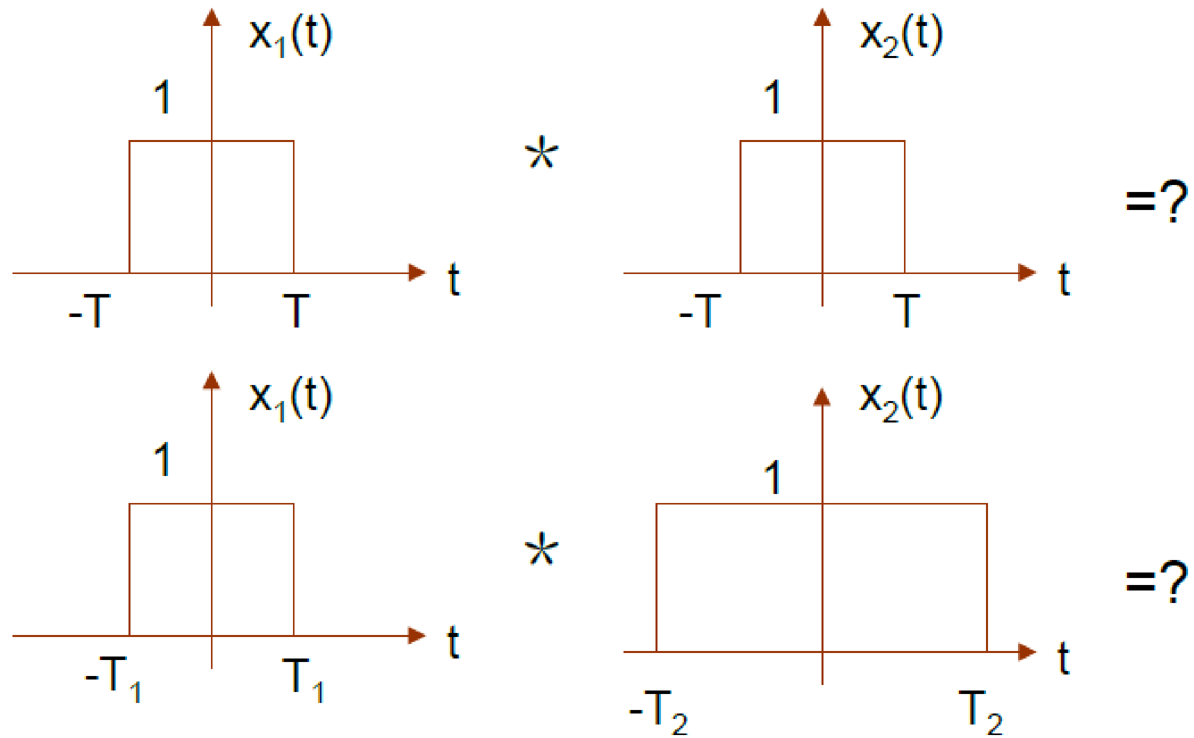
$$y(t)=\int_{0}^{T} \tau d\tau + \int_{T}^{t} T d\tau = \frac{T^2}{2} + T(t-T) = Tt - \frac{T^2}{2}$$

$$while \quad 2T \le t < 3T$$

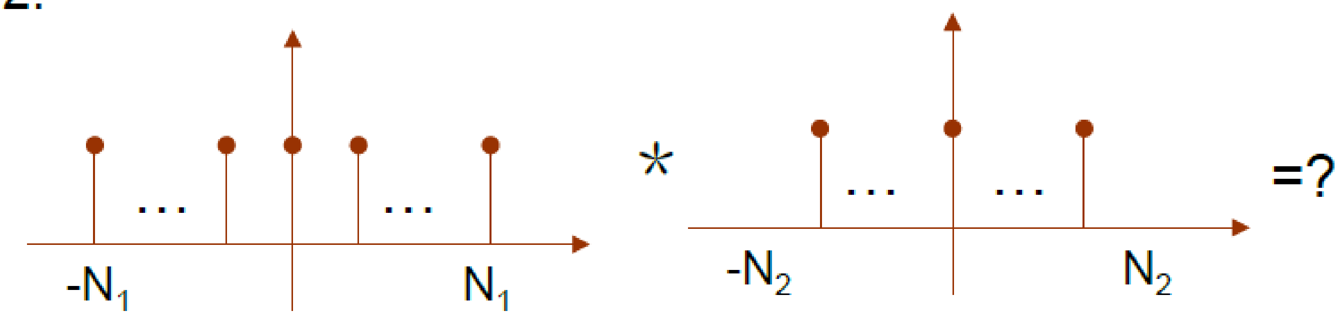
$$y(t)=\int_{0}^{T} \tau d\tau + \int_{T}^{2T} T d\tau + \int_{2T}^{t} (T-\tau)d\tau = -\frac{t^2}{2} + Tt + \frac{3T^2}{2}$$

$$others \quad y(t)=0$$

Exercise1:



Exercise2:



Supplements -- convolution

1. The Commutative Property（交换律）

$$x(t)*h(t)=h(t)*x(t)$$

2. The Distributive Property（分配律）

$$x(t)*(h_1(t)+h_2(t))=x(t)*h_1(t)+x(t)*h_2(t)$$

3. The Associative Property（结合律）

$$x(t)*(h_1(t)*h_2(t))=(x(t)*h_1(t))*h_2(t)$$

Supplements –δ function

1. Definition

$$\begin{cases} \int_{-\infty}^{+\infty} \delta(t)dt = 1 \\ \delta(t) = 0, t \neq 0 \end{cases}$$

$$\delta[n] = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases} \qquad \sum_{k=-\infty}^{\infty} \delta[k] = 1$$

2. Odd-even property (奇偶性）

$$\delta(-t) = \delta(t)$$

$$\delta[-n] = \delta[n]$$

3. The Differential and Integration Property （微积分特性 ）

$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

$$\sum_{k=-\infty}^{n} \delta[k] = u[n]$$

$$\frac{du(t)}{dt} = \delta(t)$$

$$u[n] - u[n-1] = \delta[n]$$

Supplements -δ function

4. The Shifting Property in time domain （时移特性 ）

$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$x[n] * \delta[n - m] = x[n - m]$$

5. Multiply

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$

6. Sifting property(筛选特性）

$$\int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)d\tau = x(t)$$

$$\sum_{k=-\infty}^{\infty} x[k]\delta[n - k] = x[n]$$

7. Scale property（尺度特性）

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

$$\delta[an] = \delta[n]$$

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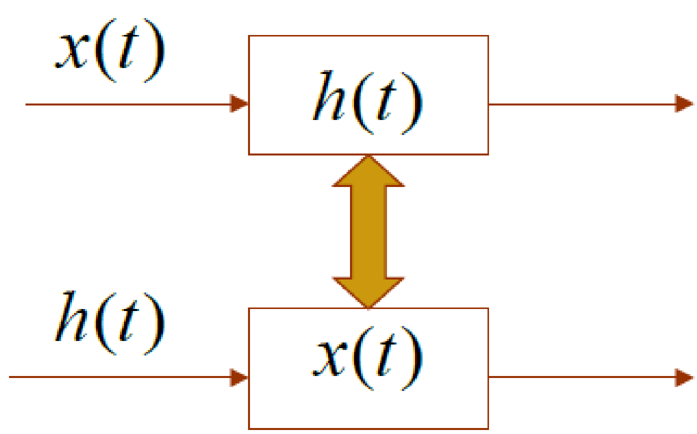
2.3.1 The Commutative Property（交换律）

$$x[n]*h[n]=h[n]*x[n]$$

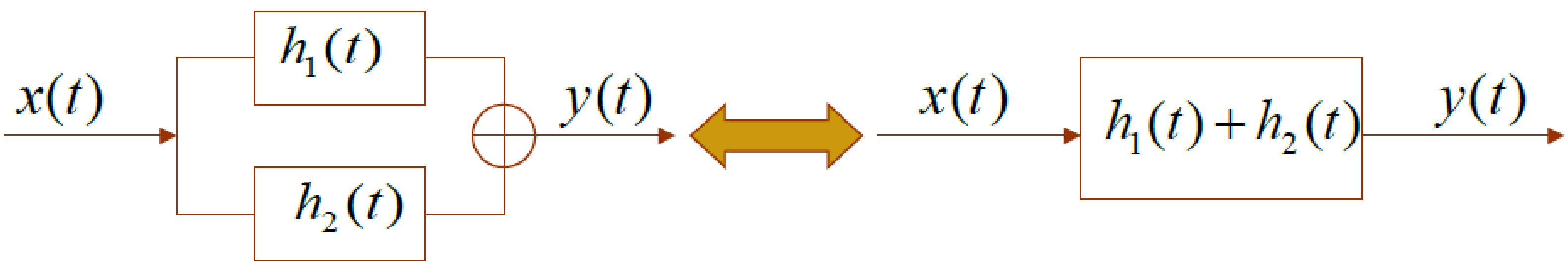
$$x(t)*h(t)=h(t)*x(t)$$



2.3.2 The Distributive Property（分配律）

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n]$$

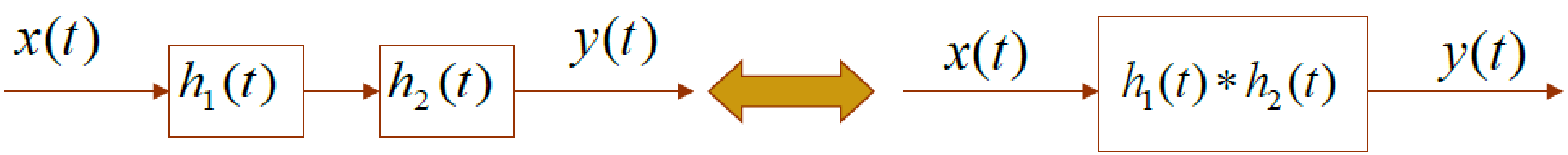
$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t)$$



2.3.3 The Associative Property （结合律）

$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n]$$

$$x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)]*h_2(t)$$



2.3.4 LTI Systems with and without Memory（有记忆和无记忆的LTI系统）

$$\because y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

According to the definition of memoryless, y[n] only depend on x[n], so

$$h[n-k] = 0, k \neq n$$

The unit impulse response of memoryless LTI:

$$h[n] = k\delta[n] \quad , \quad h[t] = k\delta[t]$$

2.3.5 Invertibility of LTI Systems（LTI系统的可逆性）

- According to the property of identical system, if h(t) is invertible and its inverse system is h1(t), then

$$h(t) * h_1(t) = \delta(t) \quad / \quad h(n) * h_1(n) = \delta(n)$$

- For example:

$$h[n] = u[n] \quad \text{——} \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]$$

$$h_1[n] = \delta[n] - \delta[n-1] \quad \text{——} \quad y[n] = x[n] - x[n-1]$$

2.3.6 Causality for LTI Systems（LTI系统的因果性）

$$\because y[n]=x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

According to the definition of causality, y[n] only depend on x[k] (k<=n), so

$$h[n-k]=0, k>n$$

i.e.

$$h(t)=0 \quad \textit{for } t<0$$

$$h[n]=0 \quad \textit{for } n<0$$

Sufficient and necessary condition

2.3.7 Stability for LTI Systems（ LTI系统的稳定性）

- *The sufficient and necessary condition for a LTI system to be stable is that its impulse response satisfies*

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty$$ *Absolutely integrable*

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$ *Absolutely summable*

Hint:

$$|x[n]| < B$$

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \le \sum_{k} |h[k]||x[n-k]| \le B\sum_{k} |h[k]|$$

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2.4.1 Linear Constant-Coefficient Differential Equations

$$\sum_{k=0}^{N} a_k \frac{\partial^k y(t)}{\partial t^k} = \sum_{k=0}^{M} b_k \frac{\partial^k x(t)}{\partial t^k}$$

- The differential equation describes the ***implicit expression*** between the input and output of the system.
- The solution of differential equations is to find the ***explicit expression*** between input and output.

- However, only a linear constant-coefficient differential/difference equation cannot specify a continuous-time system uniquely.

  Other conditions are required!

- The complete solution of a linear constant-coefficient differential/difference equation can be decomposition into:
  - ✓ Homogeneous Solution (natural response) and Particular Solution (force response)
  - ✓ Zero-Input Response and Zero-State Response
  - ✓ Transient-State Response and Steady-State Response

2.4.1.1. Homogeneous Solution (natural response) and Particular Solution (forced response) $y(t) = y_h(t) + y_p(t)$

①*homogeneous solution (齐次解):* $y_h(t)$ *satisfies*

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = 0$$

Determine the characteristic values of

$$\sum_{k=0}^{N} a_k \alpha^k = 0$$

as α_1、α_2、。。。α_N

a) if all α_i are the characteristic values of order 1, (单根)

$$y_h(t) = \sum_{i=1}^{N} A_i e^{\alpha_i t}$$

b). if there are r different characteristic values (有重根)

Suppose α_i is the k order repeated root

$$y_h(t) = \sum_{i=1}^{k} A_i t^{k-i} e^{\alpha_i t} + \sum_{j=k+1}^{N} A_j e^{\alpha_j t}$$

Notes：

1. the coefficients A_i or A_j should be determined by the auxiliary conditions simultaneously with those in the particular solution, since the condition is the auxiliary condition to determined the overall input-output relationship .

② *particular solution* (特解) $y_p(t)$

For any input signal x(t), we can get f(t) as $f(t)=\sum_{k=0}^{M} b_k x^{(k)}(t)$

Then for $y_p(t)$ is in the following forms corresponding to different forms of f(t)

f(t)	**Corresponding form of particular solution**	
t^p	$B_1 t^p + B_2 t^{p-1} + \cdots + B_p t + B_{p+1}$ **(Constant input is a special case with p=0)**	
$e^{\alpha t}$	$Be^{\alpha t}$	**α is not the characteristic value**
	$B_0 t e^{\alpha t} + B_1 e^{\alpha t}$	**α is the characteristic value with order 1**
	$B_0 t^k e^{\alpha t} + B_1 t^{k-1} e^{\alpha t} + \cdots + B_k e^{\alpha t}$	**α is the characteristic value with order k**
$\cos\beta t / \sin\beta t$	$B_1 \cos\beta t + B_2 \sin\beta t$	

③ *Determine coefficients in* $y_p(t)$ $y_h(t)$

1. Coefficients in the particular solution: substituting $y_p(t)$ in the equation

$$\sum_{k=0}^{N} a_k y^{(k)}(t) = \sum_{k=0}^{M} b_k x^{(k)}(t)$$

we can obtain all the coefficients in the particular solution, except for those items associated with $e^{\alpha t}$, where α is the characteristic value

2. Other coefficients: substituting the complete solution in the initial conditions,

$$y^{(k)}(0_+) = c_k \quad \text{for} \quad k = 0,1,\ldots,(N-1)$$

Then all coefficients in the complete solution are obtained

Notes：
1. The particular solution of a system is fully determined by the input signal;
2. As for the homogeneous solution, the form only determined by the system itself, the input signal will show its impact on the coefficients in solution

TIP

- The complete solution=

  homogeneous solution + particular solution

$$y(t) = y_h(t) + y_p(t)$$

- Natural response --- homogeneous response , determined by the characteristics of systems.

  Forced response --- particular response , determined by external incentives.

Example: $$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

$$x(t) = e^{-t}u(t) \quad y(0_+) = 2$$

Solution:

$$\because \lambda + 2 = 0 \rightarrow \lambda = \text{-}2$$

$$\therefore y_h(t) = Ae^{-2t}$$

$$\because x(t) = e^{-t}, \quad t > 0$$

$$\therefore y_p(t) = Be^{-t} \rightarrow y_p{}'(t) = -Be^{-t}$$

$$-Be^{-t} + 2Be^{-t} = Be^{-t} \rightarrow B = 1$$

$$\therefore \ y(t) = y_h(t) + y_p(t) = Ae^{-2t} + e^{-t}$$

又$y(0_+) = 2 \rightarrow A + 1 = 2 \rightarrow A = 1$

$$\therefore \ y(t) = e^{-2t} + e^{-t}. \quad t > 0$$

Exercise : $$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+2x(t)$$

$$x(t)=e^{-t}u(t) \quad y(0_+)=0 \quad y^{'}(0_+)=3$$

Notes：
The characteristic values are -1,-2, all with order 1,
The input is $x(t)=e^{-t}u(t)$

Example: $$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad (1)$$

a) $$x(t) = u(t) \quad y(0) = 0$$

b) $$x(t) = u(t+1) \quad y(0) = 0$$

Hints 1:

To calculate the particular solution and homogeneous solution in different continuous region of the input signal

Hints 2:

From the LCCDE, we can find that for input a) and b), the output and the 1st order differentiation of the output is continuous at the disrupt point of the signal.

the output
value at 0^- or 0^+

2.4.1.2. Zero-input response and Zero-states response

① *zero-input response (零输入响应)* $y_{zi}(t)$

—— system response to the non-zero initial states

—— the response is part of homogeneous solution

E.g.：in case all the characteristic values are of order 1 (所有特征根为单根)：

$$y_{zi}(t) = \sum_{k=1}^{N} A_{zik} e^{\alpha_k (t)}$$

A_{zik} could be determined by the initial states/conditions $y^{(k)}(0_-)$

②*zero-states response (零状态响应)* $y_{zs}(t)$

—— system response to the external input

—— the response includes part of the homogenous solution and particular solution

E.g.：in case all the characteristic values are of order 1 (所有特征根为单根)：

$$y_{zs}(t) = \sum_{k=1}^{N} A_{zsk} e^{\alpha_k t} + y_p(t)$$

A_{zsk} could be determined by the states changes at time 0, i.e.

$$\boxed{\{y^{(k)}(0_+) - y^{(k)}(0_-)\}}$$

③ *Complete solution*

$$y(t) = y_{zi}(t) + y_{zs}(t)$$

$$= \underbrace{\sum_k A_{zik} e^{\alpha_k t}}_{zero-input} + \underbrace{\sum_k A_{zsk} e^{\alpha_k t} + y_p(t)}_{zero-state}$$

$$= \underbrace{\sum_k A_k e^{\alpha_k t}}_{nature} + \underbrace{y_p(t)}_{forced}$$

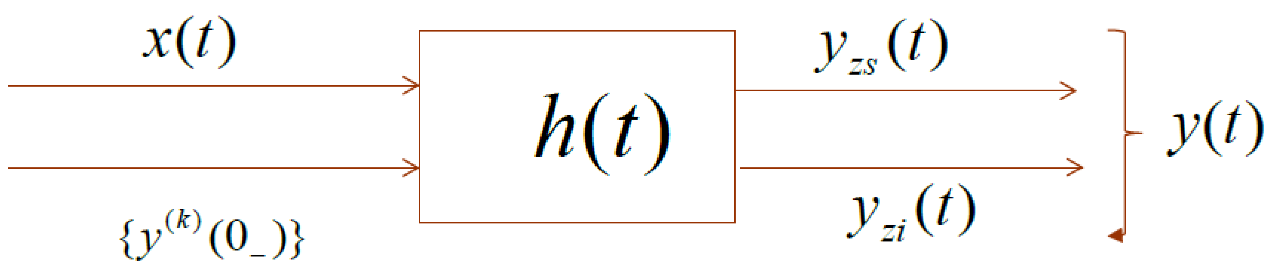
Homogeneous solution **Particular solution**

Notes: the natural response =
zero-input response + part of zero-state response

④ Linearity properties of zero-input and zero-state responses

Zero-state response is linear with the input

Zero-input response is linear with the initial state



Notes：

1. For LTI systems, the excitation and initial states can be thought of as two separate inputs.
2. When the initial condition is not zero, there is no linear relationship between the complete response of the system and the external excitation.

Example： $$\frac{dy(t)}{dt}+2y(t)=x(t)$$

$$y(0\_)=2 \quad x(t)=e^{-t}\text{u(t)}$$

Solution:

$$a+2=0 \rightarrow a=-2$$

$$\therefore y_{zi}(t)=A_{zi}e^{-2t}$$

$$y(0\_)=2 \rightarrow \quad A_{zi}=2$$

$$\therefore y_{zi}(t)=2e^{-t}$$

$$y_{zs}(t)=A_{zs}e^{-2t}+e^{-t}$$

$$y_{zs}(0_+)=y(0_+)-y(0\_)=0$$

$$\rightarrow A_{zs}=-1$$

$$\therefore y_{zs}(t)=-e^{-2t}+e^{-t}$$

$$y(t)=2e^{-2t}+(-e^{-2t}+e^{-t})=e^{-2t}+e^{-t}$$

(1) $y(0\_) = 2 \quad x(t) = e^{-t}$

$$y_{zi}(t) = 2e^{-2t} \quad y_{zs}(t) = -e^{-2t} + e^{-t} \quad y(t) = e^{-2t} + e^{-t}$$

(2) $y(0\_) = 2 \quad x(t) = 3e^{-t}$

$$y_{zi}(t) = 2e^{-2t} \quad y_{zs}(t) = 3(-e^{-2t} + e^{-t}) \quad y(t) = -e^{-2t} + 3e^{-t}$$

(3) $y(0\_) = 6 \quad x(t) = e^{-t}$

$$y_{zi}(t) = 3 \times 2e^{-2t} \quad y_{zs}(t) = -e^{-2t} + e^{-t} \quad y(t) = 5e^{-2t} + e^{-t}$$

Exercises：

For a LTI system, the input x(t) satisfies x(t)=0 for t<0, under the same initial conditions, we have

1, the overall system response to x(t) is $y_1(t) = 2e^{-t} + \cos(2t)$, for t>0

2, the overall system response to 2x(t) is $y_2(t) = e^{-t} + 2\cos(2t)$, for t>0

To determine the overall system response to 4x(t) with the same initial conditions

For zero-input response:

—Classical approach：to calculate the homogeneous solution (part)

For zero-state response

——Traditional approach：to calculate the particular solution and homogeneous solution(part)

—— Convolution integral approach： $y(t) = x(t) * h(t)$

—— Transform domain approaches： Fourier transform (frequency domain), Laplace transform (s-domain), …

2.4.1.3. Transient state response and steady state response

Transient state response: the part of the response approaching to 0, when $t \to \infty$

Steady state response: the non-zero part of the response, when $t \to \infty$

Example: $y(t) = \underbrace{-e^{-t}}_{Transient} + \underbrace{4\cos 2t}_{Steady}$

Transient state response ---- $\mathrm{Re}[\alpha_i] < 0$

Steady state response ----- $\mathrm{Re}[\alpha_i] = 0$

2.4.2 Linear Constant-Coefficient Difference Equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

1. Iterative approach
2. Time domain approaches
   - Homogeneous response + particular response
   - Zero-input and zero-status responses
   - Transient state response and steady state response

2.4.2.1. Iterative approach

Example:

$$y[n] = ay[n-1] + x[n]$$

$$x[n] = \delta[n]$$

$y[n] = 0$ when n<0

$$\because\ y[0] = ay[-1] + x[0] = a \times 0 + \delta[0] = 1$$

$$y[1] = ay[0] + x[1] = a \times 1 + \delta[1] = a$$

$$y[2] = ay[1] + x[2] = a \times a + \delta[2] = a^2$$

。 。 。

$$\therefore\ y[n] = a^n$$

Notes: Simple, but difficult to obtain the close-form response

2. Time domain approach

1. Homogeneous solution + particular solution / Natural response + forced response

① homogeneous solution (齐次解) $y_h[n]$

Determine the characteristic values of

$$\sum_{k=0}^{N} a_k \alpha^{N-k} = 0$$

as α_1、α_2、。。。α_N

a. if all α_i are the characteristic values of order 1, (单根)

$$y_h[n] = \sum_{i=1}^{N} c_i \alpha_i^n = c_1 a_1^n + c a_2^n + \cdots + c_N \alpha_N^n$$

b. if there are r different characteristic values α_i , each with the order of σ_i (r个σ_i重根α_i)

$$y_h[n] = \sum_{i=1}^{r} \sum_{k=1}^{\sigma_i} c_{ik} n^{k-1} \alpha_i^n$$

Notes：

1. the coefficients c_i or c_{ik} should be determined by the auxiliary conditions simultaneously with those in the particular solution, since the condition is the auxiliary condition to determined the overall input-output relationship .

② particular solution (特解) $y_p[n]$

For any input signal x(t), we can get f[n] as $f[n]=\sum_{k=0}^{M} b_k x[n-k]$

Then for $y_p[n]$ is in the following forms corresponding to different forms of f[n]

f[n]	Corresponding form of particular solution	
n^k	$B_1 n^k + B_2 n^{k-1} + \cdots + B_k n + B_{k+1}$	No characteristic value equals to 1
	$n^r [B_1 n^k + B_2 n^{k-1} + \cdots + B_k n + B_{k+1}]$	with characteristic value 1 of order r
α^n	$B \cdot \alpha^n$	No characteristic value equals to α
	$[B_1 n + B_2]\alpha^n$	with characteristic value α of order 1
	$[B_1 n^\gamma + B_2 n^{\gamma-1} + \cdots B_{\gamma+1}]\alpha^n$	with characteristic value α of order r
$\cos\beta n / \sin\beta n$	$B_1 \cos\beta n + B_2 \sin\beta n$	No characteristic value equals to $e^{\pm j\beta}$

③ complete Solution $y[n] = y_h[n] + y_p[n]$

Example:

$$y[n] + 2y[n-1] = x[n] - x[n-1]$$

$$x[n] = n^2, \quad n \geq 0 \qquad y[1] = -1$$

①Homogeneous Solution

$$\alpha + 2 = 0 \rightarrow \alpha = -2$$

$$\therefore y_h[n] = c \cdot (-2)^n$$

② particular solution

Substituting $x[n] = n^2$

in the right-side of the difference equation, the right-side of the equation becomes

$$f[n] = n^2 - (n-1)^2 = 2n - 1$$

Let $y_p[n] = B_1 n + B_2$

And substituting it in the difference equation to determine the coefficients B_1 and B

$$[B_1 n + B_2] + 2[B_1(n-1) + B_2] = 3B_1 n + 3B_2 - 2B_1 = 2n - 1$$

$$\therefore \begin{cases} 3B_1 = 2 \\ 3B_2 - 2B_1 = -1 \end{cases} \rightarrow \begin{cases} B_1 = \dfrac{2}{3} \\ B_2 = \dfrac{1}{9} \end{cases}$$

So $y_p[n] = \dfrac{2}{3} n + \dfrac{1}{9} B_2$

③ Complete Solution

$$y[n] = y_h[n] + y_p[1] = c \cdot (-2)^n + \frac{2}{3}n + \frac{1}{9}$$

With the condition that

$$y[1] = -1 \qquad \rightarrow c = -\frac{8}{9}$$

$$\therefore\ y[n] = \underbrace{-\frac{8}{9}(-2)^n}_{\textbf{Natural}} + \underbrace{\frac{2}{3}n + \frac{1}{9}}_{\textbf{Forced}} \qquad n \geq 0$$

As to the auxiliary conditions

When the input signal is fed into the system at $n=n_0$, define initial conditions and start condition as

- **Initial Conditions/States （起始条件）：** the system states before the signal is fed into the system, $\{y[n_0-1], y[n_0-2],\ldots, y[n_0-N]\}$
- **Start Conditions/States （初始条件）:** the first N states after the signal is fed into the system, $\{y[n_0], y[n_0+1],\ldots, y[n_0+N-1]\}$

Notes: For DT systems, we can obtain the start condition from the initial condition by means of “iterative approach”

2.4.2.2. Zero-input and zero-state responses

① Zero-input response

Example: when α_k is the characteristic value with order 1

$$y_{zi}[n] = \sum_{k=1}^{N} c_{zik} \alpha_k^n$$

c_{zik} is determined by the initial conditions

② Zero-state response

Example: when α_k is the characteristic value with order 1

$$y_{zs}[n] = \sum_{k=1}^{N} c_{zsk} \alpha_k^n + y_p[n]$$

c_{zsk} is determined by the start conditions $\{y[n_0], y[n_0+1], \ldots, y[n_0+N-1]\}$ of the ZS response, where n0 is the start time, i.e. the time the input x[n] is fed into the system.

③ $$y[n] = y_{zi}[n] + y_{zs}[n]$$
$$= \sum_k c_{zik}\alpha_k^n + \sum_k c_{zsk}\alpha_k^n + y_p[n]$$
$$= \sum_i c_i\alpha_i^n + y_p[n]$$

Notes: Natural response (homogeneous response) =
the zero-input response + part of the zero-state response

Example: $$y[n]+3y[n-1]+2y[n-2]=x[n]$$

$$x[n]=2^n u[n] \quad y[-1]=0 \quad y[-2]=\frac{1}{2}$$

Soulution:

$$\alpha^2+3\alpha+2=0 \rightarrow \alpha_1=-1 \quad \alpha_2=-1$$

(1) $$y_{zi}[n]=c_{zi1}(-1)^n+c_{zi2}(-2)^n$$

$$\begin{cases} y[-1]=0 \\ y[-2]=-1/2 \end{cases} \longrightarrow \begin{cases} c_{zi1}=1 \\ c_{zi2}=-2 \end{cases}$$

$$\therefore y_{zi}[n]=(-1)^n-2^{n+1}$$

(2) $y_{zs}[n] = c_{zs1}(-1)^n + c_{zs2}(-2)^n + B \cdot 2^n$

Substituting the particular solution $B \cdot 2^n$ in the difference equation

$$B \cdot 2^n + 3B \cdot 2^{n-1} + 2B \cdot 2^{n-2} = 2^n$$

$$(B + \frac{3}{2}B + \frac{1}{2}B) = 2^n \rightarrow B = \frac{1}{3}$$

$$\therefore y_{zs}[n] = c_{zs1}(-1)^n + c_{zs2}(-2)^n + \frac{1}{3} \cdot 2^n$$

$$\because y_{zs}[-1] = y_{zs}[-2] = 0$$

For zero-state response, the initial states are 0!
i.e. the system states before the input is fed into the system are 0!

$$\therefore y_{zs}[0] = -3y_{zs}[-1] - 2y_{zs}[-2] + x[0] = 1$$

$$y_{zs}[1] = -3y_{zs}[0] - 2y_{zs}[-1] + x[1] = -1$$

$$\longrightarrow \begin{cases} c_{zs1} = -\frac{1}{3} \\ c_{zs2} = 1 \end{cases}$$

$$\therefore y_{zs}[n] = -\frac{1}{3}(-1)^n + (-2)^n + \frac{1}{3} \cdot 2^n$$

$$\therefore y[n] = y_{zi}[n] + y_{zs}[n]$$

$$= \underbrace{(-1)^n - 2(-1)^n}_{zero-input\ response} \underbrace{-\frac{1}{3}(-1)^n + (-1)^n + \frac{1}{3}\cdot 2^n}_{zero-states\ response}$$

$$= \underbrace{\frac{2}{3}(-1)^n - (-1)^n}_{natural\ response} \underbrace{+\frac{1}{3}\cdot 2^n}_{forced\ response}$$

④ Linearity properties of zero-input and zero-state responses

Zero-state response is linear with the input

Zero-input response is linear with the initial state

Example:

$$y[n]-y[n-1]-2y[n-2]=x[n]+2x[n-2]$$

$$x[n]=u[n] \qquad y[-1]=2 \qquad y[-2]=-\frac{1}{2}$$

Determine

1. Natural response and forced response
2. Zero-input and zero-state responses

Hints: You may first calculate the zero-state response for x[n], then that for x[n-2]

Special case: to determine h[n] for a system represented by difference equations

Since h[n] is the zero-state response to the input of δ[n], the problem is equivalent to calculate the zero-state response of the system with initial conditions and input of

$$\begin{cases} x[n] = \delta[n] \\ h[n] = 0, n < 0 \end{cases} \rightarrow h[0] = \text{?}$$

例： $y[n] - 3y[n-1] + 3y[n-2] - y[n-3] = x[n]$

$$\begin{cases} x[n] = \delta[n] \\ h[n] = 0, n < 0 \end{cases} \rightarrow h[0] = 1$$

$$\because \alpha^3 - 3\alpha^2 + 3\alpha - 1 = 0$$

$(\alpha - 1)^3 = 1 \rightarrow \alpha = 1$ **is the characteristic value with order 3**

$$\therefore h[n] = c_1 n^2 + c_2 n + c_3$$

$$\begin{cases} h[0]=1 \\ h[1]=3 \\ h[2]=6 \end{cases} \rightarrow \begin{cases} c_1 = \dfrac{1}{2} \\ c_2 = \dfrac{3}{2} \\ c_3 = 1 \end{cases}$$

Notes: the coefficients could be determined by the start conditions after the impulse

$$\therefore h[n] = \frac{1}{2}(n^2 + 3n + 2), \quad n \geq 0$$

2.4.2.3. Transient state response and steady state response

Examples:

$$6y[n]-5y[n-1]+y[n-2]=x[n]$$

$$x[n]=10\cos[\frac{n\pi}{2}]\cdot u[n] \qquad y[0]=0, y[1]=1$$

$$y[n]=\underline{2\cdot(\frac{1}{2})^n-3\cdot(\frac{1}{3})^n}+\underline{\sqrt{2}\cos[\frac{n\pi}{2}-\frac{\pi}{4}]}$$

Till now, we have discussed 2 system description methods in time domain

1. h(t)/h[n]

- h(t)/h[n] could sufficiently describe a LTI system
- $y(t) = x(t) * h(t) / y[n] = x[n] * h[n]$ is the zero-state response of the system

2. Differential/difference equations

- A differential/difference equation cannot fully describe a LTI system, it needs some auxiliary conditions
- Under certain conditions, one could obtain complete response of the system

Notes: in the later of this course, we focus on the **causal LTI systems**, i.e. systems represented by difference/differential equations with initial rest conditions。

2.4.3 Linear Constant-Coefficient Differential Equations

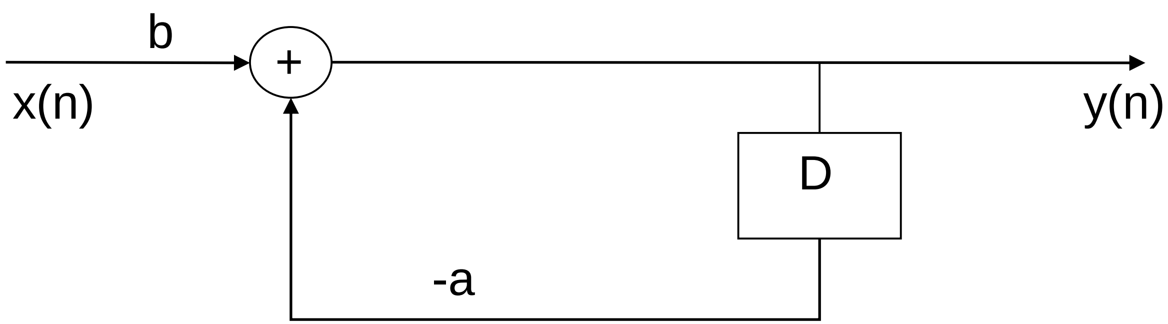
$$\sum_{k=0}^{N} a_k \frac{\partial^k y(t)}{\partial t^k} = \sum_{k=0}^{M} b_k \frac{\partial^k x(t)}{\partial t^k}$$

- The differential equation describes the ***implicit expression*** between the input and output of the system.
- The solution of differential equations is to find the ***explicit expression*** between input and output.

2.4.3 Block Diagram Representations of First-Order Systems Described by Differential and Difference Equations

1. $$y[n]+ay[n-1]=bx[n]$$

$$\rightarrow y[n]=-ay[n-1]+bx[n]$$



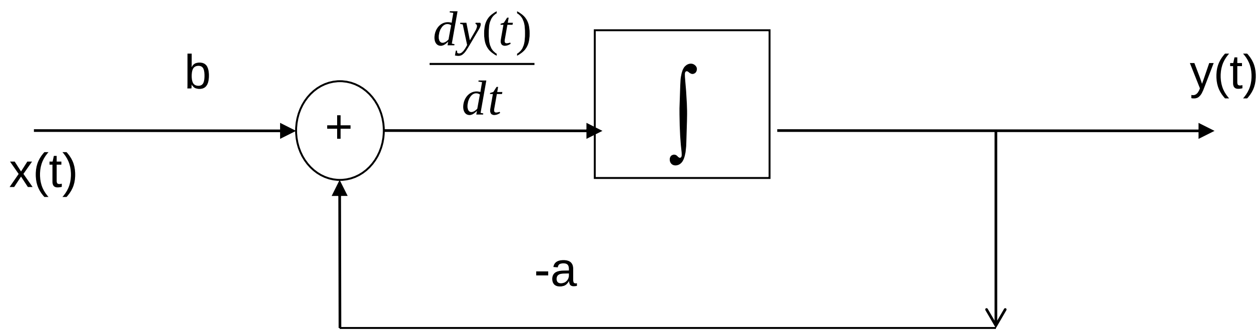
Three basic elements in block diagram:

adder(加法器), multiplier/amplifier(常系数乘法器), unit delayer(单位延时器)

2.

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\frac{dy(t)}{dt} = -ay(t) + bx(t) \rightarrow y(t) = \int_{-\infty}^{t} [-ay(\tau) + bx(\tau)] d\tau$$

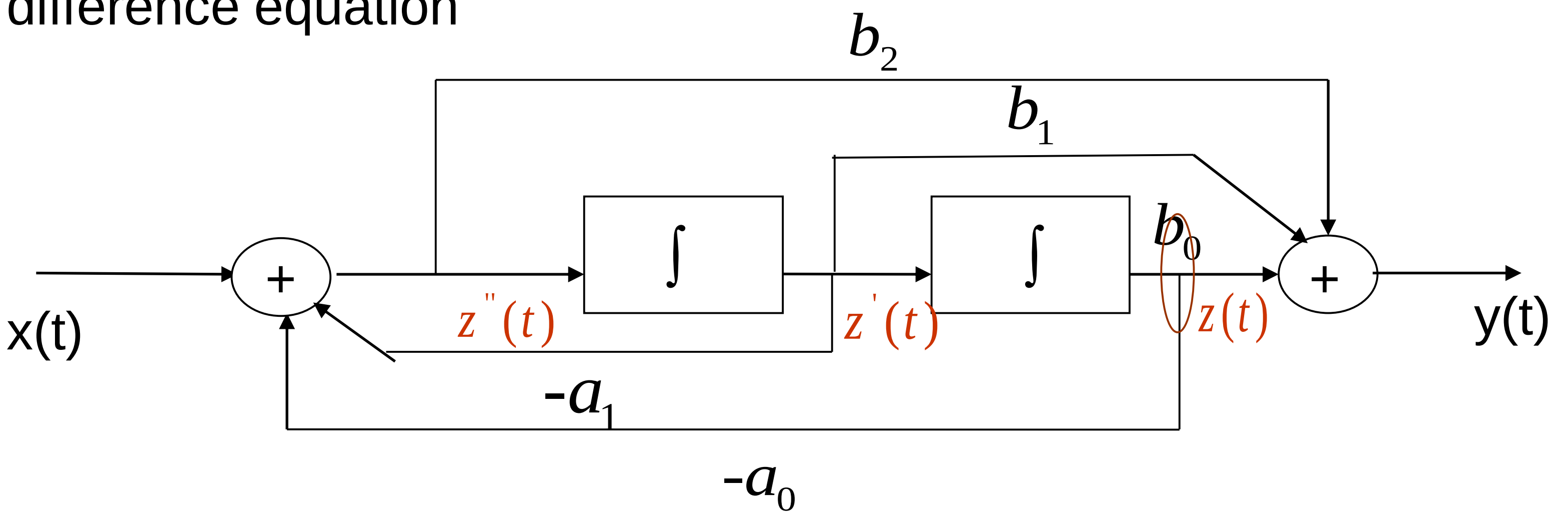


Three basic elements in block diagram:

adder(加法器), multiplier/amplifier(常系数乘法器), integrator(积分器)

Example：

To describe the following system in the form of constant-coefficient difference equation



$$y(t) = b_2 z''(t) + b_1 z'(t) + b_0 z(t)$$

$$\because y''(t) = b_2[z''(t)]'' + b_1[z'(t)]'' + b_0[z(t)]''$$

$$a_1 y'(t) = b_2[a_1 z''(t)]' + b_1[a_1 z'(t)]' + b_0[a_1 z(t)]'$$

$$a_0 y(t) = b_2[a_0 z''(t)] + b_1[a_0 z'(t)] + b_0[a_0 z(t)]$$

$$\therefore y''(t) + a_1 y'(t) + a_0 y(t)$$

$$= b_2[z''(t) + a_1 z'(t) + a_0 z(t)]''$$

$$+ b_1[z''(t) + a_1 z'(t) + a_0 z(t)]'$$

$$+ b_0[z''(t) + a_1 z'(t) + a_0 z(t)]$$

又 $z''(t) = x(t) - a_1 z'(t) - a_0 z(t)$

$\rightarrow z''(t) + a_1 z'(t) + a_0 z(t) = x(t)$

$$\therefore y''(t) + a_1 y'(t) + a_0 y(t) = b_2 x''(t) + b_1 x'(t) + b_0 x(t)$$

Homework

- BASIC PROBLEMS WITH ANSWER:
  2.1, 2.4, 2.8, 2.11, 2.19
- BASIC PROBLEMS:
  2.24, 2.40, 2.46, 2.32, 2.33

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Q & A

Many Thanks